<u>Proposition</u> $f: E \mapsto E'$. f is continuous at x_0.
$\Leftrightarrow$ $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.

<u>proof</u> $\Rightarrow$) Let $\varepsilon > 0$. since f is continuous at x_0, $\exists \delta > 0$, s.t. $f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0))$

Since $x_n \to x_0$, $\exists N > 0$, $n > N \Rightarrow x_n \in B_\delta(x_0)$.

$\Rightarrow f(x_n) \in B_\varepsilon(f(x_0))$

$\Leftarrow$) By contradiction. Assume f is not continuous at x_0, $\exists \varepsilon > 0$, s.t. $\forall \delta > 0$

$\exists x_\delta \in B_\delta(x_0)$ but $f(x_\delta) \notin B_\varepsilon(f(x_0))$.

Now, we can construct a sequence x_n, s.t.

$$x_n \in B_{\frac{1}{n}}(x_0) \text{ s.t. } f(x_n) \notin B_\varepsilon(f(x_0))$$

then $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$.

<u>Proposition</u> $f: E \mapsto \mathbb{R}$, $g: E \mapsto \mathbb{R}$. f and g are continuous at x_0, then

(a) $f+g$ is continuous at x_0

(b) $f\cdot g$ is continuous at x_0

(c) If $f(x_0)\neq 0$, then $1/f$ is continuous at x_0.

proof (c) let $Z=\{x\in E : f(x)=0\}$.

so $g(x)=\frac{1}{f(x)}$. $g: E-Z \mapsto \mathbb{R}$

Let $\varepsilon>0$, since f is continuous at x_0.

$\exists\, \delta_1>0$, s.t. $x\in B_{\delta_1}(x_0) \Rightarrow f(x)\in B_{\frac{|f(x_0)|}{2}}(f(x_0))$.

i.e., $|f(x)-f(x_0)| < \left|\frac{f(x_0)}{2}\right|$

$$\Rightarrow |f(x)| \geq |f(x_0)| - |f(x)-f(x_0)|$$

$$> \frac{|f(x_0)|}{2}$$

then $|g(x)-g(x_0)| = |1/f(x) - 1/f(x_0)|$

$$= \frac{|f(x_0)-f(x)|}{|f(x_0)|\cdot|f(x)|}$$

$$\leq \frac{2}{|f(x_0)|}\cdot|f(x)-f(x_0)|.$$

Now, select δ_2 s.t. $x \in B_{\delta_2}(x_0) \Rightarrow$
$f(x) \in B_{\varepsilon \cdot |f(x_0)|^2/2}(x_0)$.

Then, if $\delta = \min\{\delta_1, \delta_2\}$, then
$d(x, x_0) < \delta \Rightarrow$

$$|g(x) - g(x_0)| < \frac{2}{|f(x_0)|^2} |f(x) - f(x_0)|$$
$$< \frac{2}{|f(x_0)|^2} \cdot \frac{\varepsilon |f(x_0)|^2}{2}$$
$$= \varepsilon.$$

<u>Recall</u>: $f: E \mapsto E'$, f is continuous $\Leftrightarrow$
$\forall\, U' \subset E'$ open, $f(U')$ is open in E.

<u>Example</u>: E is any set, d is the discrete distance in E, $d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$

$f: E \to E'$ is continuous.

Any point is (E, d) is open, so all sets (union of points) are open. Thus, any function $f: E \mapsto E'$ is continuous.

<u>Proposition</u> $f: E \mapsto E'$, $A \subset E$. A is compact.
f is continuous $\Longrightarrow$ $f(A)$ is compact.

<u>proof</u>: Let U_i' open for all $i \in I$, s.t.

$$f(A) = \bigcup_{i \in I} U_i'.$$

Since f is continuous, $f^{-1}(U')$ is open $\forall i \in I$.

Since $f(A) \subset \bigcup_{i\in I} U_i'$, $A \subset f^{-1}(f(A)) \subset \bigcup_{i \in I} f(U_i')$

Since A is compact, $\exists\, i_1, \dots, i_n \in I$, s.t.

$$A \subset \bigcup_{j=1}^{n} f^{-1}(U_{i_j}')$$

So, $f(A) \subset \bigcup_{j=1}^{n} f(f^{-1}(U_{i_j}')) \subset \bigcup_{j=1}^{n} U_{i_j}'$.

Note, $A \subset f^{-1}(f(A))$ and $f(f^{-1}(B)) \subset B$.

for any function f and sets A and B.

Proposition $f: E \mapsto \mathbb{R}^n$, $x_0 \in E$.

$f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$

where $f_i : E \mapsto \mathbb{R}$.

f is continuous at $x_0 \Leftrightarrow f_i$ is continuous at x_0

for all $i = 1, \cdots, n$.

proof $\Rightarrow$) Let $\varepsilon > 0$, $i \in \{1, \cdots, n\}$.

$|f_i(x) - f_i(x_0)| \le d(f(x), f(x_0)) = \sqrt{\sum_{i=1}^{n} (f_i(x) - f(x_0))^2}$

select $\delta > 0$ s.t. $f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0))$

then $d(x, x_0) < \delta \Rightarrow |f_i(x) - f_i(x_0)| < \varepsilon$.

$\Leftarrow$) Let $\varepsilon > 0$, selet δ_i, s.t.

$f_i(B_{\delta_i}(x_0)) \subset B_{\frac{\varepsilon}{\sqrt{n}}}(f_i(x_0))$

So, $d(f(x), f(x_0)) \leq \sqrt{n} \max_{1 \leq i \leq n} |f_i(x) - f(x_0)|$

$$< \sqrt{n} \cdot \varepsilon/\sqrt{n} = \varepsilon$$

if $d(x, x_0) < \delta = \min_{1 \leq i \leq n} \{\delta_i\}$.

<u>Example</u> ① $f: \mathbb{R} \mapsto \mathbb{R}$, $f(x) = -x^2$

② $f(x) = \frac{-1}{(x^2+1)}$

<u>Definition</u> $f: E \mapsto \mathbb{R}$, f attains its maximum at x_0 if $f(x) \leq f(x_0)$ for all $x \in E$.

<u>Theorem</u> : E compact, $f: E \mapsto \mathbb{R}$, f continuous then f attains its maximum.

~~proof~~ Since E is compact, f is continuous

$f(E)$ is also compact.

So, $f(E)$ is closed and bounded.

Since $f(E) \subset \mathbb{R}$, let $a = \text{l.u.b}\, f(E)$

since $f(E)$ is closed, $a \in f(E)$.

Thus, $\exists\ x_0 \in E$, s.t. $f(x_0) = a$.

so, $f(x) \leq f(x_0) = a \quad \forall\ x \in E$.